Bernstein's Theorem

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Here we prove a fascinating result of Bernstein: if a function $f(x)$ is smooth on an (open) interval having non-negative derivatives of all orders, then it is in fact, real analytic.

Example: $f(x) = e^x$ satisfies the hypothesis on the entire real line.

Notation: Let I be the interval in which $f(x)$ is smooth, x_0 be a point in I and $[x_0, x_0 + R] \subset I$. Also set $X = x_0 + R$.

The proof uses *Taylor's formula with integral remainder* which says that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

then

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t) \cdot (x - t)^n dt.$$

When $n = 0$, this is just the Fundamental Theorem of Calculus. For higher n, this can be shown by repeated integration by parts.

Now we proceed to the proof of Bernstein's theorem. Two remarks: (1) Since $f^{(n+1)}$ is non-negative, $f^{(n)}$ is clearly non-decreasing. (2) For any $x \in [x_0, x_0{+}R)$, $f(x) \geq R_n(x) \geq 0$.

Changing variables $t \to x_0 + s(x - x_0)$, we obtain

$$\begin{aligned} R_n(x) &= \frac{(x - x_0)^{n+1}}{n!}\int_0^1 f^{(n+1)}(x_0 + (x - x_0)s) \cdot (1 - s)^n ds, \\ &\leq \frac{(x - x_0)^{n+1}}{n!}\int_0^1 f^{(n+1)}(x_0 + (X - x_0)s) \cdot (1 - s)^n ds. \end{aligned}$$

Notice that the RHS resembles $R_n(X)$. Upon substituting, we find:

$$R_n(x) \leq \Big(\frac{x - x_0}{X - x_0}\Big)^{n+1} R_n(X) \leq \Big(\frac{x - x_0}{X - x_0}\Big)^{n+1} f(X).$$

Taking $n \to \infty$, we find that $R_n(x) \to 0$ in $[x_0, x_0 + R)$, i.e $f(x)$ is real analytic there.

Remark: In fact, we have shown more: if we let x_0 be the midpoint of I, then the power series expansion of $f(x)$ at x_0 converges in I.