Annihilating Measures

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The technique of duality and annihilating functionals is very powerful. In the finite-dimensional setting, it looks like this: Let $W \subset V$ be vector spaces. We are interested to know when $W = V$. It isn't if and only if exists a non-trivial linear functional φ which vanishes on W. Such a linear functional will be called an *annihilating functional.* Suppose we are given $v \in V$ and we wish to find out whether v is in W. It is if and only if it is annihilated by all functionals which annihilate W. In this paper, we will use this idea in infinite dimensions to study polynomial and rational approximation in one variable.

Sketch: Müntz-Szász

Müntz and Szász answered the following question: for which sequence of positive real numbers $\lambda_1, \lambda_2, \ldots,$ are the linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \ldots$ dense in $C([0,1])$? If $\lambda_i = i$, then linear combinations are just polynomials, and the answer by Weierstrass theorem is *yes.* To answer this question, we let W to be the closure of the linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \ldots$ and see if there are non-trivial annihilating measures for W.

Suppose μ annihilates W. Then, we can form $F(z) = \int_0^1 t^z d\mu$ which is a bounded holomorphic function in the half plane $\Re(z) > -1 + \epsilon$ (to see that it is indeed holomorphic, use Morera and Fubini theorems). Furthermore, F has zeros at $0, \lambda_1, \lambda_2, \ldots$. While for arbitrary holomorphic functions, this does not pose restrictions, for bounded holomorphic functions, there is a *Blaschke condition* which requires $\sum 1/\lambda_i = \infty$. Amazingly, the condition $\sum 1/\lambda_i = \infty$ is also necessary for $W = C([0,1])$: we can form a bounded holomorphic function with the right zeros and go about representing it as $\int_0^1 t^z d\mu$ for some measure μ (there a bit of lies in the last statement, you can't use any bounded holomorphic function with the right zeros, it needs to satisfy some growth condition). In this example, the idea of annihilating measures reduced our question to that of understanding zero sets of bounded holomorphic functions. Details may be found in [3].

Measures which annihilate polynomials

Now, I will give another proof of the classical Weierstrass theorem. Suppose μ is a measure supported on $[0,1]$ which annihilates polynomials $P(\zeta)$. We can form the *Cauchy transform* of μ:

$$\hat{\mu}(z) = \int \frac{d\mu_\zeta}{\zeta - z}. \tag{1}$$

It is easy to see that the Cauchy transform converges absolutely almost everywhere and is holomorphic off the support of μ.

(Exercise: apply Fubini's theorem to show that $\int \frac{d\mu_\zeta}{|\zeta - z|}$ is in L^1_{loc})

In our case, it is holomorphic on $\mathbb{C} \setminus [0,1]$. In fact, $\hat{\mu}(\zeta)$ is 0 there. For $|z|$ large, we argue as follows:

$$\hat{\mu}(z) = \int \frac{d\mu_\zeta}{\zeta - z} = \int \Big(\sum z^{-n-1} \zeta^n \Big) d\mu_\zeta = 0.$$

(Notice that the sum is absolutely convergent). For $|z| \le 1$ but outside of $[0,1]$, we see that $\hat{\mu}(z) = 0$ by analytic continuation. Hence, $\hat{\mu}(z) = 0$ almost everywhere. I claim that this allows us to conclude that $\mu = 0$.

For this purpose, I will use the *Cauchy's integral formula for compactly supported C^1 functions:*

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g / \partial \bar{\zeta}}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}. \tag{2}$$

This follows from *Cauchy's integral formula for C^1 functions*:

$$g(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{g(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Omega} \frac{\partial g / \partial \bar{\zeta}}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

(If $g(z)$ is holomorphic, then the second term vanishes and we get the usual Cauchy's integral formula).

To see that $\mu = 0$, it suffices to check that $\int g d\mu = 0$ for all C^1 functions g. But by Fubini,

$$\int g(z) d\mu_z = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g}{\partial \bar{\zeta}} \cdot \hat{\mu}(\zeta) \, d\zeta \wedge d\bar{\zeta} = 0. \tag{3}$$

Incidentally, the converse is also true: the fact that the Cauchy transform of μ vanishes on the unbounded component of the complement of X implies that μ annihilates polynomials.

More generally, we can ask: for which sets X in the complex plane, are polynomials dense in continuous functions? In the above proof, we used the fact that the complement of $[0,1]$ is connected and that $[0,1]$ has 2-dimensional Lebesgue measure 0. The first assumption is important: for example if $X = S^1 = \{z : |z| = 1\}$ is the unit circle, then $1/z$ cannot be approximated by polynomials (exercise). The second assumption; however, can be removed by the following theorem (see [1] for a proof):

Theorem. Let $\Omega = X^c$. It can be shown that if $\hat{\mu}(z) = 0$ on Ω, then it is also 0 on points $z \in \overline{\Omega}$ where the Cauchy transform converges absolutely (which happens almost everywhere).

Measures which annihilate rational functions

I will now characterize the annihilating measures of $\mathcal{R}(X)$. Clearly, if μ is an annihilating measure, then $\hat{\mu}(z)$ vanishes off X. Conversely, if $\hat{\mu}(z)$ vanishes off X, then

$$\int \frac{d\mu_\zeta}{(\zeta - z)^n} \approx \Big(\frac{\partial}{\partial z}\Big)^{n-1} \int \frac{d\mu_\zeta}{\zeta - z} = 0$$

for $z \notin X$. By partial fractions, μ annihilates rational functions and thus their uniform limits.

Suppose now X is a compact subset in the complex plane and g is a function holomorphic on the interior of X which extends C^1 up to the boundary. *Claim: g is uniformly approximated by rational functions.* For this purpose, extend g to be a C^1 function with compact support in the complex plane.

To check that g is a rational function, we need to see that it is annihilated by all measures which annihilate $\mathcal{R}(X)$. This follows from equation (3) and the facts $\hat{\mu}(\zeta) = 0$ off X and $\frac{\partial g}{\partial \bar{\zeta}} = 0$ on X (by the Cauchy-Riemann equations and continuity).

In particular, this proves the theorems of Runge:

Theorem (Runge for Polynomials)**.** *Suppose X is a compact subset of the complex plane. Then, every function which is holomorphic on some neighbourhood of X is uniform if and only if the complement of X is connected.*

Theorem (Runge for Rational Functions)**.** *Suppose X is a compact subset of the complex plane. Then, every function which is holomorphic on some neighbourhood of X is uniform is the uniform limit of rational functions.*

In Runge's theorem for polynomials, we need only to assume that the function is holomorphic on the interior of X and extends continuously up to the boundary (in our notation, we only need it lie in $\mathcal{A}(X)$). This is known as Mergelyan's theorem. The analogue for rational functions is true for finitely connected domains, but false in general. The precise conditions for $\mathcal{R}(X) = \mathcal{A}(X)$ were given by Vitushkin in terms of "continuous analytic capacity".

Bishop's Locality Theorem

This section is to devoted to showing the fantastic theorem of Bishop which says that being in $\mathcal{R}(X)$ is a local property. More precisely:

Theorem (Bishop)**.** *Let X be a compact set in the complex plane. A continuous function $f(x)$ on X lies in $\mathcal{R}(X)$ if every point $x \in X$ has a neighbourhood U_x such that $f|_{\overline{U_x}}$ lies in $R(\overline{U_x} \cap X)$.*

Exercise: for φ smooth and compactly supported, verify the calculation:

$$\varphi\hat{\mu} = \widehat{\varphi\mu + \sigma} \qquad \text{where} \qquad \sigma = \frac{1}{2\pi i} \cdot \frac{\partial \varphi}{\partial \zeta} \cdot \hat{\mu}(\zeta)\, d\zeta \wedge d\bar{\zeta}. \tag{4}$$

Since X is compact, it is covered by finitely many neighbourhoods $\{U_{x_i}\}$ which we shall just call $\{U_i\}$. Suppose μ annihilates $\mathcal{R}(X)$. We need to show that μ annihilates f.

To prove Bishop's locality theorem, we consider a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$. If we set $\nu_i = \varphi_i\mu + \sigma_i$, then supp $\nu_i \subset U_i \cap X$, supp $\hat{\nu}_i \subset U_i \cap X$ and $\sum \nu_i = \mu$.

To see that μ annihilates f, it clearly suffices to check that each ν_i annihilates f, or more precisely, that it annihilates $f|_{U_i \cap X}$. But ν_i *does* annihilate $f|_{U_i \cap X}$ as its annihilates rational functions on $U_i \cap X$ (its Cauchy transform vanishes off $U_i \cap X$).

References

[1] L. Carleson, *Mergelyan's Theorem on Uniform Polynomial Approximation*, Math. Scand., 1964.

[2] T. Gamelin, *Uniform Algebras*, AMS, 1984.

[3] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1986.

[4] L. Zalcman, *Analytic Capacity and Rational Approximation*, Springer, 1968.