What is Analytic Capacity?

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Introduction

Everyone knows the removable singularity theorem: namely, that a bounded holomorphic function defined on the punctured disk $D \setminus \{0\}$ extends to a holomorphic function on the entire disk. What other (compact) sets K can be removed in this way besides the origin?

The proof of the removable singularity theorem extends verbatim for sets of Hausdorff dimension less than 1 or even sets of Hausdorff dimension 1, but 1-dimensional Hausdorff measure 0.

If on the other hand K has Hausdorff dimension $\alpha > 1$, one can find a measure μ supported on K for which the *Cauchy integral*

$$\hat{\mu}(z) = \int_{\mathbb{C}} \frac{d\mu_\zeta}{\zeta - z}$$

yields a bounded holomorphic function off K.

To see this, recall that $H_\beta(E) > 0$ if and only if there exists a probability measure supported on K such that $\mu(D(x,r)) \le Cr^\beta$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Pick $\beta \in (1, \alpha]$ to ensure that $H_\beta(K) > 0$ (if $H_\alpha(K) > 0$, we can just take $\beta = \alpha$). Thus exists a Borel probability measure μ supported on K satisfying $\mu(B(x,r)) \le Cr^\beta$ for all $x \in \mathbb{C}$, $r > 0$. Then, the Cauchy integral is bounded:

$$\begin{aligned} |f(z)| &\le \int_{|\zeta - z| \ge 1} d\mu + \sum_j \int_{2^{-j} \ge |\zeta - z| > 2^{-j-1}} |z - \zeta|^{-1} d\mu(\zeta) \\ &\le 1 + \sum_{j=0}^{\infty} 2^{j+1} \mu(B(z, 2^{-j})). \end{aligned}$$

It remains to check that $f(z)$ is non-constant. Clearly $f(\infty) = 0$ and $f'(\infty) = -\mu(E)$.

Above of course, $f(\infty) = \lim_{z\to\infty} f(z)$ and $f'(\infty) = \lim_{z\to\infty} zf(z)$ (the differentiation is taken with respect to the coordinate at infinity, so $f'(\infty) \ne \lim_{z\to\infty} f'(z)$).

What happens when Hausdorff dimension is 1?

Unfortunately, when the Hausdorff dimension is exactly 1, things are not so clear. It is not too hard to show that subsets of $C^{1+\epsilon}$ are removable if and only if they have positive 1-dimensional measure. Denjoy conjectured that the same is true for subsets of rectifiable curves. This is also true, but much harder to prove.

However, some Hausdorff dimension 1 sets with positive 1-dimensional measure do have analytic capacity 0. The first example has been found by Vitushkin, but later, a simpler one has been found by Garnett (e.g the linear four corners Cantor set).

Vitushkin conjectured that sets K of analytic capacity 0 were precisely sets which were *purely unrectifiable*, that is for all rectifiable curves Γ, $H_1(K \cap \Gamma) = 0$. David proved this to be true when $H_1(K)$ was finite. The problem of finding a geometric characterization of removable sets when $H_1(K) = \infty$ is still elusive.

While Hausdorff dimension does not determine removable sets for bounded holomorphic functions, Dolzhenko proved that functions analytic off K with Hölder exponent $0 < \mu < 1$ are constant if and only if $H_{1+\mu}(K) = 0$.

Analytic Capacity

Let K be a subset of the Riemann Sphere S^2 and let Ω be the exterior component of $S^2 \setminus K$. Define the *analytic capacity* of K to be the

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H(\Omega), f(\infty) = 0, \ ||f||_\infty \le 1\}.$$

Notice that if $H^\infty(\Omega)$ contains non-constant functions if and only if $\gamma(K) > 0$. Indeed, if $f(z) = a_0 + \frac{a_k}{z^k} + ...$ is in $H^\infty(\Omega)$, then $f(z)z^{k-1}$ additionally has non-zero derivative at infinity.

A simple normal family argument shows that the supremum is attained by some function f. This function turns out to be (essentially) unique and is called the *Ahlfors function*.

Clearly, γ is monotone, i.e if $K \supset K'$, then $\gamma(K) \ge \gamma(K')$. A normal family argument shows that if $K_1 \supset K_2 \supset \dots$ descend to K, then $\gamma(K) = \lim \gamma(K_i)$.

In the question posed in the introduction, we wanted to produce non-constant bounded holomorphic functions on $D \setminus K$ instead of $S^2 \setminus K$. However, it turns out that if there exist non-constant bounded holomorphic functions of $D \setminus K$, then exist non-constant bounded holomorphic functions on $S^2 \setminus K$ as well. This follows from the fact a holomorphic function on $D \setminus K$ can be written as the difference of a holomorphic function on D and a holomorphic function on $S^2 \setminus K$.

Assume that K is connected but not a point and suppose that $g : \Omega \to D$ is the Riemann map satisfying $g(\infty) = 0, g'(\infty) > 0$. Then $\gamma(K) = g'(\infty)$. Indeed, g is an admissible function, so $\gamma(K) \geq g(\infty)$. Conversely, if f is another admissible function, then $F = f \circ g^{-1}$ is a map from the unit disk to itself. Applying the Schwarz lemma, we see that $|F'(0)| \leq 1$, i.e $\left|\frac{f'(\infty)}{g'(\infty)}\right| \leq 1$, so $|f'(\infty)| \leq |g'(\infty)|$.

From the above, we can determine that the analytic capacity of a disk is given by its radius and that the analytic capacity of a line segment is a quarter of its length. The fact that these numbers coincide with the harmonic capacities is not merely a coincidence. In fact, the *harmonic capacity* $\text{Cap}(K)$ is given by the same supremum, but over multi-valued analytic functions f with single-valued $|f|$. Thus, for K with simply-connected Ω, $\text{Cap}(K) = \gamma(K)$, whereas in general $\text{Cap}(K) \geq \gamma(K)$.

In fact, analytic capacity and harmonic capacity also coincide for measurable subsets of the real line (both are equal to the quarter of the 1-dimensional measure).

Inequalities for Analytic Capacity

If K is connected then $\frac{\text{diam } K}{4} \leq \gamma(K) \leq \text{diam } K$. The equality case occurs for a line segment. Proof uses the *Koebe quarter theorem.*

Another inequality due to Pommerenke: $\gamma(K) \geq \sqrt{\frac{\text{Area } K}{\pi}}$. Here, equality holds for disks. Proof uses the *Ahlfors-Beurling inequality* $\int_K \frac{d\lambda_z}{z-\zeta} \geq \sqrt{\pi A}$.

Analytic capacity can estimate the size of functions: if $f \in H^\infty(\Omega)$ satisfies $f(\infty) = 0$ then

$$|f(z)| < \frac{\gamma(K)}{d(z,K)} \cdot ||f||_\infty.$$

We can also estimate the derivative: for $f \in H^\infty(\Omega)$,

$$|f'(z)| \leq \frac{2\gamma(K)}{d(z,K)(d(z,K) - \gamma(K))} \cdot ||f||_\infty.$$

Rational Approximation

Suppose $X \subset \mathbb{C}$ is compact. Let $\mathcal{R}(X)$ be the uniform closure of rational functions on X having poles off X. Clearly $\mathcal{R}(X) \subset \mathcal{C}(X)$ is a closed subspace.

Call $x \in X$ a *peak point* for $\mathcal{R}(X)$ if there is a function $f \in \mathcal{R}(X)$ satisfying $f(x) = 1$ while $|f(y)| < 1$ for $y \neq x$.

Given $x \in X$, we have the functional $\mathrm{ev}_x : \mathcal{R}(X) \to \mathbb{C}$ of evaluation at x. By the Hahn-Banach theorem, this functional can be represented by (possibly several) probability measures supported on X. These of course include the Dirac mass δ_x. A theorem of Bishop says that δ_x is the unique representing measure if and only if x is a peak point for $\mathcal{R}(X)$.

Bishop went on to show that $\mathcal{R}(X) = \mathcal{C}(X)$ if and only if every point of X is a peak point of $\mathcal{R}(X)$. Thus, peak points are interesting in understanding rational approximation.

Melnikov gave the following criterion for $x \in X$ to be a peak point of $\mathcal{R}(X)$: namely, we need that

$$\sum_{n=0}^{\infty} 2^n \gamma_n(x) = \infty$$

where $\gamma_n = \gamma(X^c \cap \{z : 2^{-n-1} \leq |z - x| \leq 2^{-n}\})$.

References

[1] M. Christ, *Lectures on Singular Integral Operators*, AMS, 1990.

[2] T. Gamelin, *Uniform Algebras*, AMS, 1984.

[3] J. Garnett, *Analytic Capacity and Measure*, Springer, 1973.

[4] H. Pajot, *Analytic Capacity, Rectifiability, Menger Curvature, and Cauchy Integral*, Springer, 2003.

[5] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[6] A. G. Vitushkin, *The Analytic Capacity of Sets in Problems of Approximation Theory*, Russian Mathematical Surveys, Vol 22(6), p. 139–200, 1967.

[7] T. Wolff, *Lectures on Harmonic Analysis*, AMS, 2003.

[8] E. Stout, *Polynomial Convexity*, Birkhäuser Boston, 2007.

[9] L. Zalcman, *Analytic Capacity and Rational Approximation*, Springer, 1968.